An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 42 (2001) 941–951
www.elsevier.nl/locate/camwa

Estimates of Correlation Decay in Auto/Endomorphisms of the n-Torus

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(Received September 1999; revised and accepted October 2000)

Abstract—We apply spectral methods to estimate the correlation decay of smooth and analytic observables for the algebraic automorphisms and endomorphisms of the n-torus, under conditions of hyperbolicity or pure expansivity. The exponential and superexponential estimates achieved generalize and improve analogous results already known about this subject and may provide useful reliability tests for the numerical calculation of correlations of more general maps.

Keywords—Correlations, Spectral techniques, Toral algebraic auto- and endomorphisms.

1. INTRODUCTION

In the present work, spectral methods are applied to the estimate of correlation decay in algebraic auto- and endomorphisms of the n-torus. Nowadays a large amount of information is available about this subject. Hölder continuous observables are well known to exhibit exponential decay of correlations for any hyperbolic algebraic automorphism, a result whose proof follows from the introduction of suitably defined Markov partitions and from the approximation of observables by piecewise constant functions on Markov cylinders (symbolic dynamics techniques [1,2]). More recently, Lind [3] has shown that exponential decay of correlations occurs for Lipschitz continuous observables when the hyperbolicity condition is abandoned and ergodic quasi-hyperbolic algebraic automorphisms are considered. His proof relies on general bounds to correlation decay for toral characters and on some results of symbolic dynamics, in a weaker form than that applicable to hyperbolic automorphisms, which remain valid when one or more eigenvalues of the automorphism lie on the unit circle (but they are not roots of unity).

In spite of their generality, these statements are open to improvements. The decay rate provided by symbolic dynamics is not explicitly related to the generating matrix of the hyperbolic automorphism. In the two-dimensional case, such a relation is rigorously established [4,5], but

0898-1221/01/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(01)00210-3

the result can be typically affected by the choice of the partition [6,7]. For instance, denoted with λ the eigenvalue of the associated matrix with modulus greater than 1, if the Markov partition is chosen by following Adler and Weiss' prescriptions (see [8] and references therein), all of the Markov cylinders have a self-mixing rate which turns out to be exactly $|\lambda|^{-2}$. On the contrary, whenever the generating Markov partition is achieved by prolonging both sides of the fixed-point stable and unstable manifolds, most Markov cylinders show a smaller mixing rate $|\lambda|^{-1}$ [6,7]. Furthermore, the symbolic dynamics technique of approximating Hölder continuous observables by means of piecewise constant functions on Markov cylinders completely ignores any information about the regularity of the observable itself, like smoothness and analyticity. Bounds due to Cary and Crawford [9] for the Cat Map [10] demonstrate that the correlation decay is faster for observables of greater smoothness. The spectral method applied there can be easily extended to any hyperbolic algebraic automorphism of the 2-torus, but the extension to more general endomorphisms of the n-torus is nontrivial. The same problems arise in the quasi-hyperbolic case, where decay rates are not explicitly characterized in terms of the map parameters and smoothness/analyticity of the observables not involved in the final estimates.

The basic idea which allows spectral techniques to be applied to the analysis of correlation decay is a very simple characterization of strong mixing. This states that, given an arbitrary complete orthonormal set $\mathcal{S}$ of $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$, a dynamical system $(T, \mathbb{T}^n, \mathcal{B}, \mu)$ is mixing if and only if correlations between any pair of vectors in $\mathcal{S}$ converge to zero in the limit $s \to \infty$ [2,11]. Under suitable conditions, information about the rate of correlation decay for vectors of the orthonormal base can be usefully applied to achieve estimates of the correlation decay for various classes of observables. This work provides explicit bounds to correlations of smooth and analytic observables for three classes of maps on $\mathbb{T}^n$: hyperbolic automorphisms, endomorphisms with hyperbolic tangent map, and purely expanding endomorphisms. The correlation decay in the analytic case is shown to be superexponential. Although toral algebraic auto/endomorphisms are not of direct relevance from a physical point of view, their relative simplicity makes the previous estimates useful as a reliability test for any general algorithm for the calculation of correlations of more general maps.

2. ALGEBRAIC AUTOMORPHISMS AND ENDOMORPHISMS OF THE n-TORUS $\mathbb{T}^n$.

We preliminarily recall some basic definitions and introduce the notations which will be used from now on. With $\mathbb{T}^n$, we denote the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$, parametrized by the unit cube $[0,1)^n$. As usual, $\mathbb{T}^n$ is intended to be endowed with the Lebesgue-Haar probability measure μ on the σ-field $\mathcal{B}$ of Borel sets in $[0,1)^n$. We refer to $\mathbb{R}^n$ as the covering space of the torus. The covering map $x' = x \bmod [0,1)^n$ maps any $x \in \mathbb{R}^n$ onto its only equivalent element within the unit cube. In the following, we will find it convenient to give $\mathbb{R}^n$ a Banach space structure. Depending on the context, we will use various norms and, in particular, the usual Euclidean one, $\|x\| = [\sum_{i=1}^n |x_i|^2]^{1/2}$ and that defined by $\|x\|_d = \sum_{i=1}^n |x_i|$, where x_i, $i = 1, 2, \dots, n$ stand for the components of any vector $x \in \mathbb{R}^n$ with respect to the canonical base. Obviously, owing to the equivalence of norms on a finite-dimensional linear space, two positive constants Λ_d^- and Λ_d^+ will exist such that $\Lambda_d^- \|x\|_d \le \|x\| \le \Lambda_d^+ \|x\|_d$, $\forall x \in \mathbb{R}^n$.

With the above definitions, let $[M]$ be any nonsingular $n \times n$ matrix with integer entries and consider the linear invertible transformation M of $\mathbb{R}^n$ onto itself having $[M]$ as the representative matrix with respect to the canonical base. We discuss toral maps T of the form $T(x) = M(x) \bmod [0,1)^n$, $\forall x \in \mathbb{T}^n$, which always preserve the Haar-Lebesgue measure on $\mathbb{T}^n$. $[M]$ is known as the associated (or generating) matrix of T.

Whenever $[M] \in SL(\mathbb{Z}, n)$, simple algebraic arguments show that T is a one-to-one map of $\mathbb{T}^n$ onto itself and that the probability measure μ is invariant for both T and T^{-1}. T is called an algebraic toral automorphism on $\mathbb{T}^n$. Some properties of T are directly related to the spectrum of $[M]$. In particular [8], ergodicity of T with respect to μ occurs if and only if no eigenvalue of $[M]$ is a root of the unity, whereas hyperbolicity [2] means absence of eigenvalues on the unit circle. Of course, hyperbolicity implies ergodicity, but any ergodic toral automorphism is necessarily hyperbolic when $n \leq 4$ [12]. Hyperbolicity of T on $\mathbb{T}^n$ is also equivalent to hyperbolicity of the linear mapping M on the covering plane $\mathbb{R}^n$ [2], and in that case the adjoint operator $\tilde{M}$ of M is hyperbolic as well. Let us focus our attention on $\tilde{M}$, which will actually be involved in the estimates below. Hyperbolicity of $\tilde{M}$ means that there exist two nontrivial linear subspaces of $\mathbb{R}^n$, $\mathbf{E}^s$, and $\mathbf{E}^u$, and a positive constant $\nu < 1$ satisfying the following properties:

(i) $\mathbf{E}^s \bigoplus \mathbf{E}^u = \mathbb{R}^n$;
(ii) $\tilde{M}(\mathbf{E}^s) = \tilde{M}^{-1}(\mathbf{E}^s) = \mathbf{E}^s$; $\tilde{M}(\mathbf{E}^u) = \tilde{M}^{-1}(\mathbf{E}^u) = \mathbf{E}^u$;
(iii) $k \in \mathbf{E}^s \Rightarrow \|\tilde{M}^p k\| \leq \nu^p \|k\|, \forall p \in \mathbb{N}$; $k \in \mathbf{E}^u \Rightarrow \|\tilde{M}^{-p} k\| \leq \nu^p \|k\|, \forall p \in \mathbb{N}$.

The constant ν can be easily related to the spectrum of $[M]$ [13]. $\mathbf{E}^s$ is known as the stable space, whereas $\mathbf{E}^u$ is the so-called unstable space. Conditions (ii) and (iii) lead to the further bounds $\|\tilde{M}^{-p}k\| \geq \nu^{-p}\|k\|$, $\forall\, k \in \mathbf{E}^s$, $p \in \mathbb{N}$, and $\|\tilde{M}^p k\| \geq \nu^{-p}\|k\|$, $\forall\, k \in \mathbf{E}^u$, $p \in \mathbb{N}$. Moreover, the decomposition (i) allows another norm on $\mathbb{R}^n$, $\|\cdot\|_E$, to be introduced which will be useful later. Since for every $k \in \mathbb{R}^n$, there is a unique vector $k_s \in \mathbf{E}^s$ and a unique $k_u \in \mathbf{E}^u$ such that $k = k_s + k_u$, the relationship below defines the desired norm:

$$\|k\|_E = \|k_s\| + \|k_u\| . \tag{2.1}$$

Finally, equivalence of $\|\cdot\|_E$ and $\|\cdot\|$ ensures the existence of constants $\Lambda_E^-, \Lambda_E^+ > 0$ such that $\Lambda_E^-\|k\|_E \leq \|k\| \leq \Lambda_E^+\|k\|_E$, $\forall\, k \in \mathbb{R}^n$. Although any ergodic toral automorphism is also strong mixing, so that for $n > 4$, there are nonhyperbolic mixing automorphisms, our discussion about correlation decay will be confined to hyperbolic automorphisms.

In contrast, as a simple algebraic investigation shows, the case $|\det[M]| = d \neq 1$ corresponds to a d-to-one map T of $\mathbb{T}^n$ onto itself which still preserves the Lebesgue measure and which will be referred to as an algebraic toral endomorphism. Also, in this noninvertible hypothesis, T satisfies a mixing property if and only if its associated matrix $[M]$ has no root of unity as an eigenvalue (see the Appendix for a proof). Nevertheless, we will deal with the following cases only:

(a) algebraic toral endomorphisms whose tangent map M is hyperbolic on $\mathbb{R}^n$;
(b) purely expanding algebraic toral endomorphisms.

The endomorphism T has a hyperbolic tangent map if and only if all of the eigenvalues of the associated matrix $[M]$ lie outside the unit circle, but there are eigenvalues λ_+ and λ_- satisfying $|\lambda_+| > 1$ and $|\lambda_-| < 1$. Whenever any eigenvalue of $[M]$ has modulus greater than 1, we say that the endomorphism is purely expanding. Of course, both classes of endomorphisms are mixing.

2.1. Smooth and Analytic Observables on $\mathbb{T}^n$. Correlations

According to the general definitions, an observable is any real (or possibly complex) valued square-integrable function of the torus $\mathbb{T}^n$, i.e., any function of the linear space $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ endowed with the scalar product

$$\langle f \mid g\rangle = \int_{\mathbb{T}^n} \overline{f(x)} g(x)\, d\mu(x), \qquad \forall\, f,\, g \in L^2(\mathbb{T}^n, \mathcal{B}, \mu), \tag{2.1.1}$$

and with the induced L^2-norm $\|f\|_2 = (\langle f \mid f\rangle)^{1/2}$. The mean value of $f \in L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ can then be written as $\langle 1 \mid f\rangle$ by posing $1 \in L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ such that $1(x) = 1$ for μ-almost every $x \in \mathbb{T}^n$. We denote with $a \cdot b$ the usual inner product of vectors $a, b \in \mathbb{R}^n$, i.e., the sum $\sum_{i=1}^n a_i b_i$ where $a_i \in \mathbb{R}$ stands for the i^{th} component of a with respect to the canonical base. A complete orthonormal set of $L^2(\mathbb{T}^n, \mathcal{B}, \mu)$ is given by the toral characters $e_k(x) = e^{i2\pi k\cdot x}$, $k \in \mathbb{Z}^n$, $x \in \mathbb{T}^n$, so that any observable f can be expanded into the Fourier series $f(x) = \sum_{k\in Z^n} c_k(f) e_k(x)$, convergent with respect to the L^2-norm. Smooth or analytic observables can be regarded as periodic functions on the covering space $\mathbb{R}^n$, of period 1 on each variable x_i. The Dirichlet Theorem implies the Fourier series to be convergent not only in the L^2-norm, but also pointwise on $\mathbb{T}^n$. In both cases, the regularity of f ensures a fast decay of Fourier coefficients as $\|k\| \to \infty$. More precisely, it is well known that any analytic f on $\mathbb{T}^n$ admits constants $\alpha, \beta > 0$ such that $|c_k(f)| \leq \alpha e^{-\beta\|k\|}$, $\forall\, k \in \mathbb{Z}^n$, leading to an exponential decay of the Fourier spectrum. Analogously, given a smooth (say $C^q(\mathbb{T}^n, \mathbb{C})$, $q \geq 1$) observable, there exists a nonnegative real multisequence $A(k)$, $k \in \mathbb{Z}^n$, satisfying $|c_k(f)| \leq A(k)\|k\|^{-q}$, $\forall\, k \in \mathbb{Z}^n \setminus \{0\}$ and

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} A(k)^2 \leq \left(\frac{\Lambda_d^+}{2\pi}\right)^{2q} \left[\sum_{q_1+\cdots+q_n=q} \left(\frac{q!}{q_1!q_2!\ldots q_n!}\right)^2\right] \sum_{q_1+\cdots+q_n=q} \left\|\frac{\partial^q f}{\partial x_1^{q_1}..\partial x_n^{q_n}}\right\|_2^2. \tag{2.1.2}$$

The Fourier spectrum is said to obey a power-decay law.

Our goal is to achieve estimates of correlations of smooth and analytic f, g,

$$C_s(f,g) = \langle f \mid U^s g\rangle - \overline{\langle 1 \mid f\rangle}\langle 1 \mid g\rangle, \qquad s \in \mathbb{N}, \tag{2.1.3}$$

where U stands for the Koopman operator associated to T, defined by $(Uf)(x) = f(T(x))$, $\forall\, x \in \mathbb{T}^n$, $f \in L^2(\mathbb{T}^n, \mathcal{B}, \mu)$. Owing to the polar identity, it is enough to investigate correlations of the form $C_s(f,f)$, known as the autocorrelations of the observable f.

2.2. Correlation Decay for Toral Characters

The explicit computation of correlations is a very difficult task from an analytical and a numerical point of view, and can be performed successfully in some special cases only [4–6,14]. The simplest result concerns correlations between vectors of the Fourier orthonormal base and is easily achieved by noting that the associated Koopman operator of an algebraic toral auto- or endomorphism maps the lattice $\mathbb{Z}^n$ onto itself. We have in fact, $\forall\, k \in \mathbb{Z}^n$, $x \in \mathbb{T}^n$, and $s \in \mathbb{N}$, the equality $(U^s e_k)(x) = e_k(T^s(x)) = e^{i2\pi k\cdot T^s(x)}$, which in the covering space also reads $(U^s e_k)(x) = e^{i2\pi k\cdot M^s x} = e^{i2\pi \tilde{M}^s k\cdot x}$, on having introduced the adjoint operator $\tilde{M}$ of M. As a consequence, for every $h, k \in \mathbb{Z}^n$, we get

$$\langle e_h \mid U^s e_k\rangle = \int_{[0,1)^n} e^{-i2\pi h\cdot x} e^{i2\pi \tilde{M}^s k\cdot x}\, d\mu(x) = \delta_{h,\tilde{M}^s k}, \tag{2.2.1}$$

with $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$, otherwise, for any $a, b \in \mathbb{Z}^n$. Equation (2.2.1) allows estimates of the correlation decay of regular observables to be deduced from the dynamical properties of the linear mapping $\tilde{M}$ on $\mathbb{Z}^n$.

2.3. Preliminary Estimates

Let f be an arbitrary observable and $s \in \mathbb{N}$. By the continuity of the scalar product and of the (unitary) Koopman operator with respect to the L^2-norm, we can write

$$\langle f \mid U^s f\rangle = \left\langle \sum_{h\in\mathbb{Z}^n} c_h(f) e_h \mid U^s \left[\sum_{k\in\mathbb{Z}^n} c_k(f) e_k\right]\right\rangle = \sum_{h,k\in\mathbb{Z}^n} \overline{c_h(f)} c_h(f) \langle e_h \mid U^s e_k\rangle, \tag{2.3.1}$$

and inserting (2.2.1), we obtain

$$\langle f \mid U^s f\rangle = \sum_{h,k\in\mathbb{Z}^n} \overline{c_h(f)} c_k(f) \delta_{h,\tilde{M}^s k} = \sum_{k\in\mathbb{Z}^n} \overline{c_{\tilde{M}^s k}(f)} c_k(f). \tag{2.3.2}$$

On the other hand, there also holds $\langle f \mid 1\rangle\langle 1 \mid f\rangle = \langle f \mid e_0\rangle\langle e_0 \mid f\rangle = \overline{c_0(f)} c_0(f)$ and the correlation reduces to $C_s(f,f) = \langle f \mid U^s f\rangle - \langle f \mid 1\rangle\langle 1 \mid f\rangle = \sum_{k\in Z^n\setminus\{0\}} \overline{c_{\tilde{M}^s k}(f)} c_k(f)$. The fundamental upper bound to autocorrelations will be then $|C_s(f,f)| \le \sum_{k\in Z^n\setminus\{0\}} |c_{\tilde{M}^s k}(f)|$ $|c_k(f)|$, provided that the series on the right-hand side converges.

If the spectrum decays exponentially, the previous bound becomes

$$|C_s(f,f)| \le \alpha^2 \sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{-\beta(\|\tilde{M}^s k\|+\|k\|)}, \tag{2.3.3}$$

for some constants $\alpha, \beta > 0$. In an analogous way, the estimate for the case of a power-law decay spectrum takes the form

$$|C_s(f,f)| \le \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{A(k)}{\|k\|^q} \frac{A\left(\tilde{M}^s k\right)}{\left\|\tilde{M}^s k\right\|^q}, \qquad q \ge 1. \tag{2.3.4}$$

We can drop the terms $A(k)$, $A(\tilde{M}^s k)$ and conveniently increase the exponent q, which may be very important to ensure convergence of the upper bound. Two subsequent applications of the Cauchy-Schwarz inequality allow us to write

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{A(k)}{\|k\|^q} \frac{A\left(\tilde{M}^s k\right)}{\left\|\tilde{M}^s k\right\|^q} \le G^{1/2} H^{1/4} \left[\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{\|k\|^{4q} \left\|\tilde{M}^s k\right\|^{4q}}\right]^{1/4}, \tag{2.3.5}$$

where the constants $G = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} A(k)^2$ and $H = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} A(h)^4$ are both finite because of (2.1.2). Thus, with a suitable constant $L > 0$ depending on the observable f, we deduce

$$|C_s(f,f)| \le L \left[\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{\|k\|^{4q} \left\|\tilde{M}^s k\right\|^{4q}}\right]^{1/4}. \tag{2.3.6}$$

The latter estimate can be replaced by a weaker but more satisfactory bound, which also recalls the form (2.3.3). To this end, by the equivalence of norms $\|\cdot\|_d$ and $\|\cdot\|$, we have

$$\|k\| \left\|\tilde{M}^s k\right\| \ge \left(\Lambda_d^-\right)^2 \|k\|_d \left\|\tilde{M}^s k\right\|_d = \frac{1}{4}\left(\Lambda_d^-\right)^2 2\|k\|_d 2\left\|\tilde{M}^s k\right\|_d, \tag{2.3.7}$$

and if we confine ourselves to $k \in \mathbb{Z}^n \setminus \{0\}$, and remember the inclusion $\tilde{M}(\mathbb{Z}^n \setminus \{0\}) \subseteq \mathbb{Z}^n \setminus \{0\}$, it is straightforward to verify that $2\|k\|_d$, $2\|\tilde{M}^s k\|_d \ge 2$. The subsequent lower bound $2\|k\|_d$ $2\|\tilde{M}^s k\|_d \ge 2\|k\|_d + 2\|\tilde{M}^s k\|_d$ easily follows from the inequality $xy \ge x+y$, valid $\forall\, x, y \ge 2$. Inserting within (2.3.7), we finally get

$$\|k\| \left\|\tilde{M}^s k\right\| \ge \frac{1}{4}\left(\Lambda_d^-\right)^2 \left(2\|k\|_d + 2\left\|\tilde{M}^s k\right\|_d\right) \ge \frac{1}{2\Lambda_d^+}\left(\Lambda_d^-\right)^2 \left(\|k\| + \left\|\tilde{M}^s k\right\|\right), \tag{2.3.8}$$

and the estimate (2.3.4) is replaced by

$$|C_s(f,f)| \le L \frac{(2\Lambda_d^+)^q}{(\Lambda_d^-)^{2q}} \left[\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{\left(\|k\| + \left\|\tilde{M}^s k\right\|\right)^{4q}}\right]^{1/4}. \tag{2.3.9}$$

The above discussion shows that the behaviour of $\|k\| + \|\tilde{M}^s k\|$ on $s \in \mathbb{N}$ is crucial in order to establish meaningful estimates of $C_s(f,f)$ for analytic and smooth observables.

2.4. Decay of Correlations for Hyperbolic Algebraic Automorphisms of $\mathbb{T}^n$

Suppose that the map T is a hyperbolic automorphism of $\mathbb{T}^n$. This implies, in particular, that the linear transformation M defines a one-to-one mapping of the lattice $\mathbb{Z}^n$ onto itself, and so does $\tilde{M}$. Let f be an analytic $L^2(\mathbb{T}^n,\mathcal{B},\mu)$-function, for which, therefore, (2.3.3) holds, and suppose for simplicity that the index $s\in\mathbb{N}$ of the correlation $C_s(f,f)$ is even. By introducing the change of variable $h=\tilde{M}^{s/2}k$, the bound (2.3.3) is put into the following equivalent form:

$$|C_s(f,f)|\le\alpha^2\sum_{h\in\mathbb{Z}^n\setminus\{0\}}e^{-\beta(\|\tilde{M}^{s/2}h\|+\|\tilde{M}^{-s/2}h\|)}. \tag{2.4.1}$$

Recalling the definitions and notations of Section 2.1, concerning the hyperbolic structure of $\tilde{M}$, we obtain

$$\frac{1}{\Lambda_E^-}\left(\left\|\tilde{M}^{s/2}h\right\|+\left\|\tilde{M}^{-s/2}h\right\|\right)\ge\left\|\tilde{M}^{s/2}h_s+\tilde{M}^{s/2}h_u\right\|_E+\left\|\tilde{M}^{-s/2}h_s+\tilde{M}^{-s/2}h_u\right\|_E, \tag{2.4.2}$$

but since $\tilde{M}^{s/2}h_s,\tilde{M}^{-s/2}h_s\in\mathbf{E}^s$ and $\tilde{M}^{s/2}h_u,\tilde{M}^{-s/2}h_u\in\mathbf{E}^u$, we can rewrite the right-hand side as

$$\left\|\tilde{M}^{s/2}h_s\right\|+\left\|\tilde{M}^{s/2}h_u\right\|+\left\|\tilde{M}^{-s/2}h_s\right\|+\left\|\tilde{M}^{-s/2}h_u\right\|\ge\nu^{-s/2}\frac{1}{\Lambda_E^+}\|h\|. \tag{2.4.3}$$

As a conclusion, $\|\tilde{M}^{s/2}h\|+\|\tilde{M}^{-s/2}h\|\ge\nu^{-s/2}\|h\|\Lambda_E^-/\Lambda_E^+$. The case of odd $s\in\mathbb{N}$ is treated in a completely similar way, by posing $h=\tilde{M}^{(s-1)/2}k$ within (2.3.3), and the result reads $\|\tilde{M}^{(s+1)/2}h\|+\|\tilde{M}^{-(s-1)/2}h\|\ge\nu^{-(s-1)/2}\|h\|\Lambda_E^-/\Lambda_E^+$. We now simply replace into (2.4.1) and conclude

$$|C_s(f,f)|\le\alpha^2e^{-\nu^{-\lfloor s/2\rfloor}\beta\Lambda_E^-/\Lambda_E^+}\sum_{h\in\mathbb{Z}^n\setminus\{0\}}e^{-\nu^{-\lfloor s/2\rfloor}\beta(\|h\|-1)\Lambda_E^-/\Lambda_E^+},\qquad\forall s\in\mathbb{N}, \tag{2.4.4}$$

where $\lfloor x\rfloor$ stands for the integer part of $x\in\mathbb{R}$ and the residual series is bounded uniformly on s. For $f\in C^q(\mathbb{T}^n,\mathbb{C})$ and even $s\in\mathbb{N}$, the transformation $k=\tilde{M}^{s/2}h$ puts the series on the right-hand side of (2.3.9) into the form

$$\sum_{h\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{\left(\left\|\tilde{M}^{-s/2}h\right\|+\left\|\tilde{M}^{s/2}h\right\|\right)^{4q}}, \tag{2.4.5}$$

which admits the upper bound $\nu^{2qs}(\Lambda_E^+)^{4q}(\Lambda_E^-)^{-4q}\sum_{h\in Z^n\setminus\{0\}}\|h\|^{-4q}$ due to the previous bound on $\|\tilde{M}^{-s/2}h\|+\|\tilde{M}^{s/2}h\|$. Inequality (2.3.9) then becomes

$$|C_s(f,f)|\le L\frac{(2\Lambda_d^+)^q}{(\Lambda_d^-)^{2q}}\left(\frac{\Lambda_E^+}{\Lambda_E^-}\right)^q\nu^{sq/2}\left[\sum_{h\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{\|h\|^{4q}}\right]^{1/4}, \tag{2.4.6}$$

the constant $L>$ depending only on f. The substitution $s\to s-1$ provides the analogous bound for odd s, as in the analytic case. Therefore,

$$|C_s(f,f)|\le L\frac{(2\Lambda_d^+)^q}{(\Lambda_d^-)^{2q}}\left(\frac{\Lambda_E^+}{\Lambda_E^-}\right)^q\nu^{q\lfloor s/2\rfloor}\left[\sum_{h\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{\|h\|^{4q}}\right]^{1/4},\qquad\forall s\in\mathbb{N}. \tag{2.4.7}$$

It is understood that the given estimate is meaningful provided that the residual series in (2.4.6) converges, which occurs whenever $q>n/4$.

REMARK. When the complexification of the linear operator $\tilde{M}$ can be diagonalized on $\mathbb{C}^n$, a small modification of the previous discussion provides a more specific characterization of the decay rates. For simplicity, let us denote with the same symbol $\tilde{M}$ the complexification of $\tilde{M}$. Suppose then that $\tilde{M}$ admits the base $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ of eigenvectors on $\mathbb{C}^n$ with corresponding (possibly complex or coinciding) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, none of which lies on the unit circle. Any vector $h \in \mathbb{R}^n$ will be written in a unique way as

$$h = \sum_{i=1}^{n} c_i(h) u_i, \quad c_i(h) \in \mathbb{C}, \qquad \forall i = 1, 2, \ldots, n, \tag{2.4.8}$$

and a norm $\|\cdot\|_u$ will be defined by $\|h\|_u = \sum_{i=1}^n |c_i(h)|$, equivalent to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$ according to $\Lambda_u^- \|h\|_u \leq \|h\| \leq \Lambda_u^+ \|h\|_u$, $\forall\, h \in \mathbb{R}^n$, $\Lambda_u^-, \Lambda_u^+ > 0$. We have

$$\left\|\tilde{M}^{s/2} h\right\| + \left\|\tilde{M}^{-s/2} h\right\| \geq \operatorname*{Inf}_j \left[|\lambda_j|^{s/2} + |\lambda_j|^{-s/2}\right] \frac{\Lambda_u^-}{\Lambda_u^+} \|h\|, \tag{2.4.9}$$

where $\mathrm{Inf}_j\left[\,|\lambda_j|^{s/2} + |\lambda_j|^{-s/2}\right]$ increases exponentially as $s \to +\infty$. An analogous calculation holds for odd s and provides

$$\left\|\tilde{M}^{(s+1)/2} h\right\| + \left\|\tilde{M}^{-(s-1)/2} h\right\| \geq \operatorname*{Inf}_j \left[|\lambda_j|^{1/2} \left(|\lambda_j|^{s/2} + |\lambda_j|^{-s/2}\right)\right] \frac{\Lambda_u^-}{\Lambda_u^+} \|h\| . \tag{2.4.10}$$

2.5. Decay of Correlations for Algebraic Endomorphisms of $\mathbb{T}^n$ with Hyperbolic Tangent Map

The case of algebraic endomorphism with a hyperbolic tangent map can be treated as the previous one. We only have to notice that now $\tilde{M}$ defines a transformation of $\mathbb{R}^n$ which is still one-to-one but not onto. As a consequence, $\tilde{M}\left(\mathbb{Z}^n \setminus \{0\}\right) \subset \mathbb{Z}^n \setminus \{0\}$. Consider, for instance, a correlation $C_s(f,f)$ with f analytic and even $s \in \mathbb{N}$. With the change of variables $h = \tilde{M}^{s/2} k$, which is well defined, and due to the hyperbolic structure of $\tilde{M}$, (2.3.3) writes

$$|C_s(f,f)| \leq \alpha^2 \sum_{h \in \tilde{M}^{s/2}(\mathbb{Z}^n \setminus \{0\})} e^{-\beta(\|\tilde{M}^{s/2}h\| + \|\tilde{M}^{-s/2}h\|)}. \tag{2.5.1}$$

An analogous estimate holds for odd s by introducing the change of variable $h = \tilde{M}^{(s-1)/2} k$. The same bounds on $\|\tilde{M}^{-s/2}h\| + \|\tilde{M}^{s/2}h\|$ and $\|\tilde{M}^{(s+1)/2}h\| + \|\tilde{M}^{-(s-1)/2}h\|$ established in Section 2.4 lead then to the inequality

$$|C_s(f,f)| \leq \alpha^2 e^{-\nu^{-\lfloor s/2 \rfloor} \beta \Lambda_E^- / \Lambda_E^+} \sum_{h \in \mathbb{Z}^n \setminus \{0\}} e^{-\nu^{-\lfloor s/2 \rfloor} \beta (\|h\| - 1) \Lambda_E^- / \Lambda_E^+}, \qquad \forall\, s \in \mathbb{N}. \tag{2.5.2}$$

Given an $f \in C^q(\mathbb{T}^n, \mathbb{C})$, instead of (2.4.5) we investigate the series

$$\sum_{h \in \tilde{M}^{s/2}(\mathbb{Z}^n \setminus \{0\})} \frac{1}{\left(\left\|\tilde{M}^{-s/2} h\right\| + \left\|\tilde{M}^{s/2} h\right\|\right)^{4q}}, \tag{2.5.3}$$

and deduce the bound

$$\nu^{2qs} \left(\frac{\Lambda_E^+}{\Lambda_E^-}\right)^{4q} \sum_{h \in \tilde{M}^{s/2}(\mathbb{Z}^n \setminus \{0\})} \frac{1}{\|h\|^{4q}} \leq \nu^{2qs} \left(\frac{\Lambda_E^+}{\Lambda_E^-}\right)^{4q} \sum_{h \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{\|h\|^{4q}}, \tag{2.5.4}$$

so that (2.4.6) still holds, for even s. Odd values of s and problems of convergence of the residual series are dealt with as above, and the final estimate coincides with (2.4.7).

REMARK. If the complexification of the $\tilde{M}$ can be put into a diagonal form on $\mathbb{C}^n$, the decay rates are more explicitly characterizable in the same way we have already outlined for hyperbolic automorphisms, as estimates (2.4.9) and (2.4.10) obviously extend to the present case too.

2.6. Correlation Decay for Purely Expanding Algebraic Endomorphisms of $\mathbb{T}^n$

For expanding endomorphisms, all of the eigenvalues of the adjoint operator $\tilde{M}$ have modulus greater than 1. A positive constant $\nu < 1$ will exist such that $\|\tilde{M}^{-1}k\| \leq \nu\|k\|$ and $\|\tilde{M}k\| \geq \nu^{-1}\|k\|$, $\forall\, k \in \mathbb{R}^n$, and therefore,

$$\left\|\tilde{M}^s k\right\| + \|k\| \geq \nu^{-s}\|k\| + \|k\| = \left(\nu^{-s}+1\right)\|k\| . \tag{2.6.1}$$

For each $s \in \mathbb{N}$, and with no change of variables, the upper bound (2.3.3) to the autocorrelation $C_s(f,f)$ of an analytic observable f will take the form

$$|C_s(f,f)| \leq \alpha^2 \sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{-\beta(\nu^{-s}+1)\|k\|} = \alpha^2 e^{-\beta(\nu^{-s}+1)} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} e^{-\beta(\nu^{-s}+1)(\|k\|-1)} \tag{2.6.2}$$

with the usual bounded residual series. The same argument applied to an observable $f \in C^q(\mathbb{T}^n,\mathbb{C})$ leads to the estimate

$$|C_s(f,f)| \leq L\frac{(2\Lambda_d^+)^q}{(\Lambda_d^-)^{2q}}\frac{1}{(\nu^{-s}+1)^q}\left[\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{\|k\|^{4q}}\right]^{1/4}, \qquad \forall\, s\in\mathbb{N}, \tag{2.6.3}$$

immediately derived from (2.3.9).

REMARK. Whenever the complexification of $\tilde{M}$ is diagonalizable on $\mathbb{C}^n$, we can determine a very simple relation between the expansion rate ν^{-1} and the eigenvalues of the linear operator. Indeed by using (2.4.8), we obtain

$$\left\|\tilde{M}^s h\right\| + \|h\| \geq \Lambda_u^-\left(\left\|\tilde{M}^s h\right\|_u + \|h\|_u\right) \geq \frac{\Lambda_u^-}{\Lambda_u^+}\left(\left[\operatorname*{Inf}_j|\lambda_j|\right]^s + 1\right)\|h\| , \tag{2.6.4}$$

where $\mathrm{Inf}_j|\lambda_j| > 1$.

REMARK. In the case of the Cat Map, it is interesting to compare our estimates with those described in [9]. There the decay of correlations for analytic observables turned out to be more than exponential, whereas we can state a more precise super-exponential decay law. Although not explicitly explained here, the systematic construction of observables obeying a power-decay law can be performed as well by means of lacunar Fourier series. For instance, any continuous g of the form

$$g(x) \equiv \sum_{r=-\infty, r\neq 0}^{\infty} \frac{1}{r^2} e^{i2\pi\tilde{M}^r k\cdot x}, \qquad k\in\mathbb{Z}^n\setminus\{0\}, \quad x\in\mathbb{T}^n \tag{2.6.5}$$

shows an algebraic correlation decay, according to $C_s(g,g) = (2/3)\pi^2(1/s^2) - 6/s^4$. Nevertheless, if λ denotes the eigenvalue of $\tilde{M}$ with modulus greater than one, the exponential decay rate computed in [9] for a C^q observable is $q\ln|\lambda|$, exactly twice the value $q\ln|\lambda|/2$ we find. This better result is not surprising, since it comes from a very detailed characterization of the orbits of $\tilde{M}$ on the reciprocal lattice $\mathbb{Z}^2$, followed by an ingenious, suitable rearrangement of the orbits themselves. The extension of the same arguments to other algebraic automorphisms of the 2-torus and to higher-dimensional auto- and endomorphisms is certainly nontrivial and fairly cumbersome, in spite of the simplicity and generality of the present analysis. In both cases, the spectral method clearly relates smoothness of the observable and estimated decay rate.

3. CONCLUSIONS

For hyperbolic systems, spectral methods relate smoothness of the observable to its (possibly superexponential) decay rate. Standard techniques of symbolic dynamics, by contrast, predict a decay rate which is only exponential, independent of the smoothness of the observable, and not easily related to the map parameters. Symbolic dynamics techniques are based on approximations of the observables by means of piecewise constant functions on Markov cylinders. The error introduced by this first approximation is exponentially small with respect to $s \in \mathbb{N}$ by assuming the additional requirement that observables be Hölder-continuous. C^q, $q \in \mathbb{N} \setminus \{0\}$, or C^ω observables are certainly Hölder-continuous, as Lipschitz-continuous, but the estimate of the approximation error carries no trace of such a regularity. Spectral techniques, even if by paying the price of a lesser generality, offer the twofold advantage of a rather strict relation between the estimated decay rate and the smoothness of the observables on the one hand, and between the decay rate and the system parameters on the other one. As for nonhyperbolic systems, a domain where general methods of symbolic dynamics are not available, spectral techniques can provide quite satisfactory estimates to correlation decay of analytic or sufficiently smooth observables as well. We further stress that the superexponential decay of correlations for analytic observables is a *per se* interesting result. Moreover, it is noticeable that the symbolic dynamics method has been recently implemented by a numerical tool for the computation of correlations in the hyperbolic automorphisms of the 2-torus [7]. The tool uses piecewise constant approximation on the cylinders of suitably refined Markov partitions, whose typical portrait is shown in Figure 1, to provide estimates of correlations of the following form:

$$C_s(f,f) \simeq a\lambda^{-s} + b\lambda^{-2s}. \tag{3.1}$$

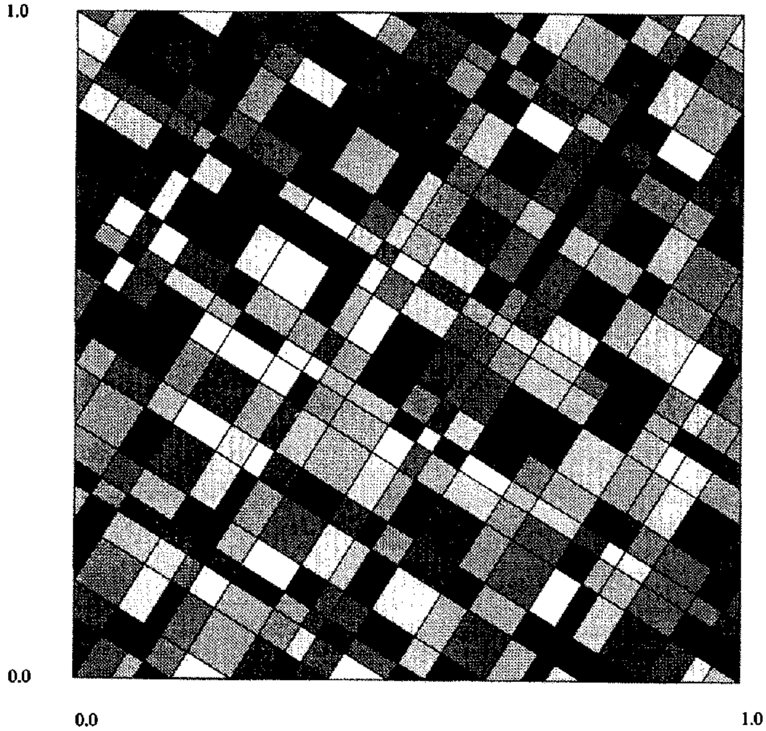

Figure 1. A refined Markov partition for the Cat Map on the 2-torus, parametrized by the unit square. Grey levels evidence different rectangles.

This relation holds after the mixing time of the partition for appropriate constants $a, b \in \mathbb{R}$, and up to an exponentially decreasing rest. Quite satisfactory results can be obtained for discontinuous, continuous but nonsmooth, and C^1 observables, whereas the accuracy is lower in the case of C^3 or analytic functions. The estimates presented here account for this failure of the algorithm, since a relation like (3.1) cannot adequately describe either a superexponential correlation decay, or correlations of the form $C_s(f,f) \sim \lambda^{-q^s}$, $q > 2$, for large s. As a general conclusion, the tight analytic bounds to correlations of nontrivial observables discussed here constitute a useful test to check the accuracy of any general algorithm for the computation of correlations.

APPENDIX

We sketch here the proof of the following.

PROPOSITION A.1. *An algebraic toral endomorphism T is mixing if and only if its associated matrix $[M]$ has no eigenvalue which is a root of unity.*

PROOF. We preliminarily observe that $[M]$, M, and $\tilde{M}$ have exactly the same spectrum. The opposite implications will be shown separately.

(i) If no eigenvalue of $[M]$ is a root of unity, then T is mixing.

Suppose that T is not mixing. This implies that there exists at least a pair of Fourier vectors e_k, e_h, $k, h \in \mathbb{Z}^n$, whose correlations $C_s(e_k, e_h) = \langle e_k \mid U^s e_h\rangle - \langle e_k \mid 1\rangle\langle 1 \mid e_h\rangle$ do not converge to zero. We can actually assume $k, h \in \mathbb{Z}^n \setminus \{0\}$, since otherwise correlation decay always occurs. According to (2.2.1), the binary sequence $(\delta_{k,\tilde{M}^s h})_{s\in N}$ does not tend to zero as $s \to \infty$, and therefore, a sequence of positive integers $(n_\ell)_{\ell\in N}$ can be found such that $n_1 < n_2 < \cdots < n_\ell < \cdots$ and $\tilde{M}^{n_\ell} h = k$, $\forall \ell \in \mathbb{N}$. In particular, we deduce $(\tilde{M}^{n_2-n_1} - \mathbb{1})\tilde{M}^{n_1} h = 0$, and conclude that $\tilde{M}^{n_1} h$ is an eigenvector of $\tilde{M}^{n_2-n_1}$ with eigenvalue 1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\tilde{M}$, there will exist $j \in \{1, 2, \ldots, n\}$ such that $\lambda_j^{n_2-n_1} = 1$ and λ_j is a root of unity, a contradiction.

(ii) If T is mixing, then no eigenvalue of $[M]$ is a root of unity.

We still prove the statement by *reductio ad absurdum*. Let $\lambda = e^{i2\pi p/q}$ be an eigenvalue of $\tilde{M}$, with $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Then the power $\tilde{M}^q$ has 1 as an eigenvalue and the equation $(\tilde{M}^q - \mathbb{1})\xi = 0$ admits a nontrivial solution $\xi \in \mathbb{R}^n$. Furthermore, since the last equation is a linear system with integer coefficients, we can certainly find a solution $\xi \in \mathbb{Z}^n \setminus \{0\}$. Therefore, $\tilde{M}^q \xi = \xi$, and consequently,

$$\tilde{M}^{mq}\xi = \xi, \qquad \forall m \in \mathbb{N}, \tag{A.1}$$

from which it follows that

$$C_{mq}(e_\xi, e_\xi) = \langle e_\xi \mid U^{mq} e_\xi\rangle - \langle e_\xi \mid 1\rangle \langle 1 \mid e_\xi\rangle = \delta_{\xi,\tilde{M}^{mq}\xi} = 1, \qquad \forall m \in \mathbb{N}. \tag{A.2}$$

As a conclusion, the autocorrelations $C_s(e_\xi, e_\xi)$ do not converge to zero for $s \to \infty$, which contradicts the mixing hypothesis on T. ∎

From the above statement, we also deduce the following proposition as a simple corollary. It implies equivalence of ergodicity and mixing property for any algebraic toral endomorphism.

PROPOSITION A.2. *An algebraic toral endomorphism T is ergodic if and only if its associated matrix $[M]$ has no eigenvalue which is a root of unity.*

PROOF. If no eigenvalue of $[M]$ is a root of unity, Proposition A.1 states that T is mixing, which implies ergodicity. We simply have to prove that when T is ergodic, the spectrum of $[M]$ does not contain roots of unity. If not, according to Proposition A.1, it would be possible to find out a vector $\xi \in \mathbb{Z}^n \setminus \{0\}$ and a positive integer q satisfying (A.1) and (A.2). For every $s \in \mathbb{N} \setminus \{mq : m \in \mathbb{N}\}$, we have generically $C_s(e_\xi, e_\xi) = \delta_{\xi,\tilde{M}^s\xi}$, and therefore, $0 \le C_s(e_\xi, e_\xi) \le 1$. As a consequence, given an integer $N \in \mathbb{N}$ large enough, we could write

$$\frac{1}{N}\sum_{s=0}^{N-1} C_s(e_\xi, e_\xi) \ge \frac{1}{N} \sum_{m=0}^{\lfloor (N-1)/q\rfloor - 1} \sum_{k=0}^{q-1} C_{mq+k}(e_\xi, e_\xi) \ge \frac{1}{N}\left\lfloor \frac{N-1}{q} \right\rfloor \xrightarrow[N\to+\infty]{} \frac{1}{q} > 0. \tag{A.3}$$

This means that the sequence of Cesaro's sums $(N^{-1}\sum_{s=0}^{N-1} C_s(e_\xi, e_\xi))_{N\in\mathbb{Z}^+}$ does not converge to zero as $N \to +\infty$, in contradiction with the ergodicity of T. The proof is complete. ∎

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